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J. Phys. A: Math., Nucl. Gen., Vol. 6, January 1973. Printed in Great Britain. © 1973

SU(3) in an O(3) basis I. Properties of shift operators

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MS received 2 February 1972, in revised form 27 July 1972

Abstract. With a view to obtaining an orthogonal solution to the state labelling problem of SU(3) in an O(3) basis, four independent operators which shift the eigenvalues of the O(3) Casimir operator L^2 are constructed. The hermiticity properties of these operators, and of certain of their products which commute with L^2, are discussed.

1. Introduction

The group SU(3) contains two distinct types of SU(2) subgroups, an example of the more familiar type being the isotopic spin subgroup of the SU(3) group of hadron physics. In this case the additional SU(3) generators transform under commutation with the SU(2) generators as two doublets and a scalar tensor representation of SU(2). For the other type of SU(2) subgroup, the additional generators transform like a single five-dimensional tensor representation. Examples of this type are the subgroup of the group of unimodular unitary (3×3) matrices consisting of the orthogonal matrices, and the subgroup of the SU(3) symmetry group of the three-dimensional harmonic oscillator generated by the angular momentum operators (Allen Baker Jr 1956, Lipkin 1965). A marked difference also arises in the decomposition of irreducible representations of SU(3) upon restriction to the two types of subgroups; in the former case representations specified by half integral l values occur (where $l(l+1)$ is the eigenvalue of the SU(2) Casimir L^2), whereas in the latter case only representations corresponding to integral l values arise (Elliot 1958a, b, Bargmann and Moshinsky 1960, 1961 and De Baenst-van den Broucke *et al* 1970). For this reason the two groups may be distinguished by the titles SU(2) and O(3), respectively.

The analysis of representations of SU(3) into irreducibles of its O(3) subgroup is considerably harder than for its SU(2) subgroup, due to the difficulty in distinguishing between different states corresponding to the same degenerate l value. Whereas the degeneracy of any l value appearing in a given SU(3) representation is already well known (Elliot 1958a, b, Bargmann and Moshinsky 1960, 1961 and De Baenst-van den Broucke *et al* 1970), so far attempts at obtaining an orthogonal specification of the corresponding states have had only limited success.

There exist two hermitian O(3) scalar operators, O_l^0 and Q_l^0 of, respectively, third and fourth orders in the group generators, and either of these, together with L^2 and the O(3) generator l_0, form a complete set of commuting hermitian operators for the irreducible SU(3) representation spaces. Mutually orthogonal states corresponding to degenerate l values can in principle therefore be obtained by choosing them to be eigenvectors of O_l^0 or Q_l^0. However, Racah (1962) reported that he and collaborators were unable to find a method of calculating the eigenvalues of any hermitian combination of these operators. He showed in particular that no such combination could have rational eigenvalues, a fact borne out in the following paper. Elliot (1958a, b) and Bargmann and Moshinsky (1960, 1961) meanwhile derived methods of distinguishing degenerate l states by means of parameters which did take on rational values, but the states they obtained were not orthogonal.

In this and the following paper we shall show that a conceptually simple and systematic, if algebraically somewhat involved, method does in fact exist for calculating the eigenvalues of O_l^0 and Q_l^0, thus in principle solving the problem of obtaining an orthogonal specification of the degenerate l states. The solution rests upon the existence of some previously unknown shift operators, $O_l^{\pm 1}$ and $O_l^{\pm 2}$, which change the l values of states upon which they act by, respectively, ± 1 and ± 2 without altering the l_0 value, and in this paper we concern ourselves with the derivation and properties of these operators. They are analogous to operators derived by Stone (1956) for the O(3) subgroup of O(4), although for that group only operators of the type $O_l^{\pm 1}$ exist. It is precisely because of the existence for SU(3) of two pairs of such operators, which give alternative ways of connecting states of different l values, that l degeneracy occurs. Starting from the maximum l state, which is annihilated by O_l^{+1} and O_l^{+2}, states of successively lower l values may be defined by the repeated action of O_l^{-1} and O_l^{-2}. Because various products of these operators, such as $O_{l+1}^{-1}O_l^{+1}$, which commute with L^2, are expressible in terms of O_l^0 and Q_l^0, the matrix elements of the latter operators can readily be obtained; it is then a straightforward matter to calculate their eigenvalues.

In § 2 we shall summarize the forms of the generators to be employed in this paper and relate them to the more familiar generators used when discussing the SU(2) subgroup of SU(3). Expressions for O_l^0 and Q_l^0 and the SU(3) invariants I_2 and I_3 will also be given. The remainder of the section is concerned with the calculation of the shift operators.

Section 3 deals with the hermiticity properties of these operators, which are somewhat complicated by the fact that $O_l^{\pm 1}$ and $O_l^{\pm 2}$ are 'one-sided' operators, that is, they act as shift operators only upon states to the right and not upon states to the left. In § 4 we give expressions in terms of O_l^0, Q_l^0 and the SU(3) invariants for those products of the shift operators which commute with L^2. The application of these operators to the solution of the state labelling problem will be treated in the following paper†.

2. Construction of the shift operators

The most commonly used choice for the generators of the group SU(3) is the one consisting of the Cartan subalgebra H_1, H_2 and its root vectors $E_{\pm\alpha}$, $E_{\pm\beta}$ and $E_{\pm\bar{\beta}}$. These are the generators used, for instance, by Baird and Biedenharn (1963) who summarize their commutation relations. H_1 and $E_{\pm\alpha}$ together generate an SU(2) subgroup; the pairs of generators $E_{\pm\beta}$ and $E_{\pm\bar{\beta}}$ both form two-dimensional tensor representations of this subgroup, and H_2 a one-dimensional representation.

SU(3) possesses two invariants I_2 and I_3 of, respectively, second and third order, whose eigenvalues serve to specify uniquely its irreducible representations. Baird and Biedenharn (1963) show that every unitary irreducible representation may be labelled by the pair of integers (p, q) satisfying $p \geqslant q \geqslant 0$ and related to I_2 and I_3 by the formulae

$$I_2 = \tfrac{1}{9}(p^2+q^2-pq+3p) \tag{1}$$

† This paper will appear in a following issue of the journal.

and

$$I_3 = \tfrac{1}{162}(p-2q)(2p+3-q)(p+q+3) \tag{2}$$

(p, q) and $(p, p-q)$ are contragredient representations.

In this paper we shall be concerned with a different form for the generators, defined in terms of the above generators by

$$\begin{aligned} l_0 &= 2\sqrt{3}H_1, \qquad l_\pm = 2\sqrt{3}(E_{\pm\bar\beta} - E_{\pm\beta}) \\ q_0 &= -6H_2, \qquad q_{\pm1} = -3\sqrt{2}(E_{\pm\beta} + E_{\pm\bar\beta}), \qquad q_{\pm2} = -6E_{\pm\alpha}. \end{aligned} \tag{3}$$

These satisfy the commutation relations

$$\begin{aligned} [l_0, l_\pm] &= \pm l_\pm, \qquad [l_+, l_-] = 2l_0 \\ [l_0, q_\mu] &= \mu q_\mu, \qquad [l_\pm, q_\mu] = \pm(-1)^\mu\{6-\mu(\mu\pm1)\}^{1/2} q_{\mu\pm1} \end{aligned} \tag{4}$$

where μ takes on the values 0, ±1, ±2, and

$$\begin{aligned} [q_0, q_{\pm1}] &= \mp 3l_\pm\sqrt{\tfrac{3}{2}}, \qquad [q_0, q_{\pm2}] = 0 \\ [q_{+1}, q_{-1}] &= 3l_0, \qquad [q_{+2}, q_{-2}] = 6l_0 \\ [q_{\pm2}, q_{\mp1}] &= \mp 3l_\pm, \qquad [q_{\pm1}, q_{\pm2}] = 0. \end{aligned} \tag{5}$$

q_0 is hermitian, whereas $q_{\pm1}$ and $q_{\pm2}$ are pairs of hermitian conjugate operators. l_0 and $l_\pm$ together generate an O(3) subgroup of SU(3), with respect to which the q_μ form a five-dimensional irreducible tensor representation.

I_2 and I_3 are given in terms of these generators by the formulae:

$$I_2 = \tfrac{1}{36}(3L^2 + q_0^2 + q_{+1}q_{-1} + q_{-1}q_{+1} + q_{+2}q_{-2} + q_{-2}q_{+2}) \tag{6}$$

$$\begin{aligned} I_3 = \frac{1}{2\times(36)^2}[&2q_0\{2q_0^2 + 3(q_{+1}q_{-1} + q_{-1}q_{+1}) - 6(q_{+2}q_{-2} + q_{-2}q_{+2}) + 9(L^2 - 3l_0^2 - 8)\} \\ &- 6\sqrt{6}(q_{+2}q_{-1}^2 + q_{-2}q_{+1}^2) + 9\sqrt{6}(l_+^2 q_{-2} - l_-^2 q_{+2}) \\ &+ 18\sqrt{6}(l_0 - 2)l_+ q_{-1} + 18\sqrt{6}(l_0 + 2)l_- q_{+1}] \end{aligned} \tag{7}$$

where $L^2 \equiv l_0(l_0 - 1) + l_+ l_-$ is the Casimir of O(3). The irreducible representations of O(3) will be labelled by l, where $l(l+1)$ is the eigenvalue of L^2.

SU(3) possesses two hermitian O(3) scalar operators (Racah 1962) of third and fourth orders in the group generators. These will play a fundamental role in the following paper in giving an orthogonal specification of degenerate l states, and are

$$O_l^0 = \sqrt{6}q_0(L^2 - 3l_0^2 - 6) + 3l_+ q_{-1}(2l_0 - 3) + 3l_- q_{+1}(2l_0 + 3) + 3(l_+^2 q_{-2} + l_-^2 q_{+2}) \tag{8}$$

and

$$\begin{aligned} Q_l^0 = {} & 2(q_0^2 + q_{+1}q_{-1} - 2q_{+2}q_{-2})(L^2 - 3l_0^2 - 6) + \sqrt{6}l_+(q_0 q_{-1} - \sqrt{6}q_{+1}q_{-2})(2l_0 - 3) \\ &+ \sqrt{6}l_-(q_0 q_{+1} - \sqrt{6}q_{-1}q_{+2})(2l_0 + 3) - 2\sqrt{6}(l_+^2 q_0 q_{-2} + l_-^2 q_0 q_{+2}) \\ &- 3(l_+^2 q_{-1}^2 + l_-^2 q_{+1}^2) + 9l_0(L^2 - 3l_0^2 - 15l_0 - 6). \end{aligned} \tag{9}$$

Although l appears in these expressions only implicitly via L^2, so that their forms are independent of the l values of states upon which they act, we shall find it convenient to suffix them with l since their eigenvalues do depend on l. Also, O_l^0 is a particular example of the l shifting operators which we are about to construct, and the forms of these do depend explicitly on l. O_l^0 and Q_l^0 do not commute and so cannot be simultaneously diagonalized except when acting on states corresponding to nondegenerate l values. The apparatus for obtaining their eigenvalues consists of the l shift operators, which we now derive.

As basis vectors for the representation (p, q) we employ the kets $|p, q; l, a_l, m\rangle$, where m is the eigenvalue of l_0 and a_l is the as yet undefined parameter needed for their unique specification, satisfying the orthonormality condition

$$\langle p, q; l, a_l, m|p, q; l', b_{l'}, m'\rangle = \delta_{l,l'}\delta_{a_l,b_{l'}}\delta_{m,m'}. \tag{10}$$

Since the shift operators will not change p and q, they will be suppressed in the following. We look for 'pure' l shift operators which leave m unchanged, whose action on the kets must therefore have the form

$$O_l|l, a_l, m\rangle \propto |l', b_{l'}, m\rangle.$$

O_l must clearly commute with l_0 and have a commutator with L^2 of the form

$$[L^2, O_l] = 2\lambda O_l. \tag{11}$$

We shall also require O_l to contain the q_μ to first order only, and therefore choose it of the form

$$O_l = \sqrt{6}q_0 + al_+q_{-1} + bl_-q_{+1} + cl_+^2q_{-2} + \mathrm{d}l_-^2q_{+2} \tag{12}$$

which, up to an overall multiplicative constant, is the most general such operator which also commutes with l_0.

Using the commutation relations (4), one obtains an expression for $[L^2, O_l]$ in terms of $q_0, l_\pm q_{\mp 1}$ and $l_\pm^2 q_{\mp 2}$, in which the coefficients depend on both l and m and are obtained by replacing L^2 and l_0 by their eigenvalues whenever they lie to the extreme right of all other operators. Use of equation (11) also gives rise to an expression for $[L^2, O_l]$ in terms of the q_0 etc, so by equating coefficients of the five operators in the two expressions we obtain five linear equations to solve for a, b, c and d. The conditions for a solution is the vanishing of the determinant of coefficients of 1, a, b, c and d, and this yields the following fifth order equation to be solved for λ:

$$\lambda(\lambda-l-1)(\lambda+l)(\lambda-2l-3)(\lambda+2l-1) = 0. \tag{13}$$

Note that this is independent of m.

The permissible values of λ are therefore $k(2l+k+1)$ where $k = 0, \pm 1, \pm 2$, and for each of these values we may solve for a, b, c and d in terms of l and m. On multiplying the resulting O_l by a common denominator and replacing m by l_0 again, we obtain five operators, the first of which is just the O_l^0 of equation (8), the other four being

$$\begin{aligned} O_l^{+1} = {} & \sqrt{6}l_0(l^2-l_0^2-3)q_0+(l_0-l-1)(2l_0+l-2)l_+q_{-1}+(l_0+l+1)(2l_0+2-l)l_-q_{+1} \\ & +(l_0-l-1)l_+^2q_{-2}+(l_0+l+1)l_-^2q_{+2} \end{aligned} \tag{14}$$

$$\begin{aligned} O_l^{-1} = {} & \sqrt{6}l_0(l^2+2l-l_0^2-2)q_0+(l_0+l)(2l_0-l-3)l_+q_{-1} \\ & +(l_0-l)(2l_0+l+3)l_-q_{+1}+(l_0+l)l_+^2q_{-2}+(l_0-l)l_-^2q_{+2} \end{aligned} \tag{15}$$

$$\begin{aligned} O_l^{+2} = {} & \sqrt{6}(l_0+l+1)(l_0-l-1)(l_0^2-l^2+4)q_0-2(l_0-l-1)(l_0+l)(l_0-l-2)l_+q_{-1} \\ & -2(l_0+l+1)(l_0-l)(l_0+l+2)l_-q_{+1}-(l_0-l-1)(l_0-l-2)l_+^2q_{-2} \\ & -(l_0+l+1)(l_0+l+2)l_-^2q_{+2} \end{aligned} \tag{16}$$

$$O_l^{-2} = \sqrt{6}(l_0+l)(l_0-l)(l_0^2-l^2-2l+3)q_0 - 2(l_0+l-1)(l_0-l-1)(l_0+l)l_+q_{-1}$$
$$-2(l_0-l+1)(l_0+l+1)(l_0-l)l_-q_{+1} - (l_0+l-1)(l_0+l)l_+^2q_{-2}$$
$$-(l_0-l+1)(l_0-l)l_-^2q_{+2}. \tag{17}$$

Since

$$[L^2, O_l^k] = k(2l+k+1)O_l^k, \qquad k = 0, \pm 1, \pm 2 \tag{18}$$

one easily deduces that

$$L^2(O_l^k|l, a_l, m\rangle) = (l+k)(l+1+k)(O_l^k|l, a_l, m\rangle)$$

so that

$$O_l^k|l, a_l, m\rangle \propto |l+k, b_{l+k}, m\rangle \tag{19}$$

thus justifying the k superscript. The l subscript is justified by the fact that, apart from O_l^0 (which is the only one of these operators which is hermitian), O_l^k depends explicitly on the l value of the state upon which it acts since l does not appear in its expression solely in the form $l(l+1)$, which could be replaced by the operator L^2.

Now, in analogy with the fact that $\langle l, m|l_\pm \propto \langle l, m\mp 1|$, it might be tempting to suppose that

$$\langle l, b_l, m|O_l^k \propto \langle l-k, a_{l-k}, m|,$$

in other words that, when acting to the left upon bras, O_l^{+i} and O_l^{-i} $(i = 1, 2)$ are, respectively, l lowering and l raising operators. This would be the case if, like $l_\pm$, O^{+i} and O^{-i} were hermitian conjugate operators; however, they are not, due to their dependence on l. Equation (18) holds only when acting to the right upon kets since, in replacing L^2 by $l(l+1)$ in the calculations of the O_l^k, it is assumed that all operators involved act to the right so L^2 has first to be placed to the extreme right of all other operators. If they were acting to the left upon bras, L^2 would have to be placed to the extreme left of all other operators before such a replacement would be permissible, and this would give rise, through commutation, to extra terms which would alter the expressions for the O_l^k. It is in fact easily verifiable by explicit calculations (see later) that $(O_l^k)^\dagger \neq O_{l+k}^{-k}$, and therefore

$$\langle l, b_l, m|O_l^k \not\propto \langle l-k, a_{l-k}, m|. \tag{20}$$

O_l^k, as given by equations (14)–(17), are therefore 'one-sided' l shift operators, shifting l by k only when acting to the right. Similarly, $(O_l^k)^\dagger$ may act only to the left upon bras:

$$\langle l, a_l, m|(O_l^k)^\dagger \propto \langle l+k, b_{l+k}, m|, \tag{21}$$

but

$$(O_l^k)^\dagger|l+k, b_{l+k}, m\rangle \not\propto |l, a_l, m\rangle. \tag{22}$$

Inspection of equations (14)–(17) shows that the shift operators exhibit some symmetry, namely that for $i = 1, 2$, O_l^{+i} and O_l^{-i} can be obtained from each other by replacing l by $-(l+1)$. The reason for this is that the eigenvalue $l(l+1)$ is unchanged by this replacement; in fact the representations of O(3) can be labelled either with $l = 0, 1, 2, \ldots,$ or with $l = -1, -2, -3, \ldots.$ Using the specification of representations in terms of positive l, O_l^{+i} increases l by i and therefore L^2 by the positive amount $i(2l+i+1)$. Suppose, instead, we specify the representations in terms of negative l; then O_l^{+i} still increases l by i, but increases the eigenvalue of L^2 by the now negative amount $i(2l+i+1)$.

Writing $l = -(l'+1)$, where l' is positive, this means that $O^{+i}_{-(l'+1)}$ decreases the eigenvalue of L^2 by $i(2l'-i+1)$, which is precisely what the operator $O^{-i}_{l'}$ does in the positive l specification. O^{+i}_l and O^{-i}_l therefore interchange their roles on passing from a specification of representations of O(3) in terms of positive l to one in terms of negative l. As we should expect, O^0_l, which leaves l unchanged, is invariant under $l \to -(l+1)$.

Finally, in this section, we observe two facts. Firstly, the requirement that the shift operators be of first order in q_μ is not too restrictive, since l shift operators of higher order in q_μ may easily be constructed from them; for instance $O^{-1}_{l+2}O^{+2}_l$ is a second order operator in q_μ shifting l by $+1$, whereas $O^{+2}_{l+2}O^{+2}_l$ and $O^{+1}_{l+3}O^{+2}_{l+1}O^{+1}_l$ are operators of, respectively, second and third orders in the q_μ which shift l by $+4$. Secondly, in the construction of the shift operators, no use was made of commutators of the form $[q_\mu, q_\nu]$; only the fact that the q_μ transform according to an irreducible five-dimensional tensor representation of O(3) was employed. This strongly suggests that similar shift operators may be constructed from arbitrary tensor representations, and we hope to consider this problem in a later paper.

3. Hermiticity properties of the shift operators

Several formulae, which will prove essential in the following paper in the calculation of matrix elements of the shift operators, may be derived once the constants $\alpha_{i,l}$, $i = 1, 2$ appearing in the equation

$$\langle l, a_l, m|(O^{+i}_l)^\dagger|l+i, b_{l+i}, m\rangle = \alpha_{i,l}\langle l, a_l, m|O^{-i}_{l+i}|l+i, b_{l+i}, m\rangle \tag{23}$$

are known, so we now calculate them.

Taking the hermitian conjugate of equation (14), we obtain

$$\begin{aligned}(O^{+1}_l)^\dagger = {} & \surd 6 l_0(l^2+2l-l_0^2+1)q_0+(l_0+l+1)(2l_0-l-2)l_+q_{-1} \\ & +(l_0-l-1)(2l_0+l+2)l_-q_{+1}+(l_0+l+1)l_+^2q_{-2}+(l_0-l-1)l_-^2q_{+2}.\end{aligned} \tag{24}$$

Comparison of this with equation (15) shows immediately that $(O^{+1}_l)^\dagger \neq O^{-1}_{l+1}$. Defining the difference operator E by

$$(O^{+1}_l)^\dagger = \alpha_{1,l}O^{-1}_{l+1}+E$$

and taking matrix elements between $\langle l, a_l, m|$ and $|l+1, b_{l+1}, m\rangle$, we obtain

$$\langle l, a_l, m|E|l+1, b_{l+1}, m\rangle = 0,$$

which on substitution of the easily calculated expression for E becomes

$$\begin{aligned}0 = {} & m\{(l^2+2l+1-m^2)-\alpha_{1,l}(l^2+4l+1-m^2)\}\surd 6\langle l, a_l, m|q_0|l+1, b_{l+1}, m\rangle \\ & +(m+l+1)\{(2m-l-2)-\alpha_{1,l}(2m-l-4)\}\langle l, a_l, m|l_+q_{-1}|l+1, b_{l+1}, m\rangle \\ & +(m-l-1)\{(2m+l+2)-\alpha_{1,l}(2m+l+4)\}\langle l, a_l, m|l_-q_{+1}|l+1, b_{l+1}, m\rangle \\ & +(m+l+1)(1-\alpha_{1,l})\langle l, a_l, m|l_+^2q_{-2}|l+1, b_{l+1}, m\rangle \\ & +(m-l-1)(1-\alpha_{1,l})\langle l, a_l, m|l_-^2q_{+2}|l+1, b_{l+1}, m\rangle.\end{aligned} \tag{25}$$

The various matrix elements may be calculated in terms of reduced matrix elements using the formula (Edmonds 1957)

$$\langle l, a_l, m|q_\mu|l', b_{l'}, m'\rangle = t_\mu(-1)^{l-m}\begin{pmatrix} l & 2 & l' \\ -m & \mu & m' \end{pmatrix}(l, a_l\|q\|l', b_{l'})$$

where $\begin{pmatrix} l & 2 & l' \\ -m & \mu & m' \end{pmatrix}$ is a 3-j symbol and $t_\mu = +1$ for $\mu = -1, \pm 2$, and $t_\mu = -1$ for $\mu = 0, 1$.

On substituting these matrix elements in equation (25) and dividing through by the common reduced matrix element $(l, a_l\|q\|l+1, b_{l+1})$, one obtains an equation for $\alpha_{1,l}$ whose solution is

$$\alpha_{1,l} = \frac{2l+1}{2l+3}. \tag{26}$$

In a precisely analogous manner, one obtains

$$\alpha_{2,l} = \frac{2l+1}{2l+5}. \tag{27}$$

Now from the O_l^k we may construct various operators which commute with L^2, the ones with which we shall be concerned being of one of the types $O_{l+k}^{-k}O_l^{+k}$, $O_{l\pm 1}^{\mp 1}O_{l\pm 2}^{\mp 1}O_l^{\pm 2}$, $O_{l\mp 1}^{\pm 1}O_{l\pm 1}^{\mp 2}O_l^{\pm 1}$, and $O_{l\mp 2}^{\pm 2}O_{l\mp 1}^{\mp 1}O_l^{\mp 1}$. That these do commute with L^2 follows from the fact that the sums of the superscripts in their constituent operators all vanish; for instance

$$O_{l+1}^{-1}O_l^{+1}|l, a_l, m\rangle \propto O_{l+1}^{-1}|l+1, b_{l+1}, m\rangle \propto |l, a_l', m\rangle$$

and so $O_{l+1}^{-1}O_l^{+1}$ leaves l unchanged; there is, of course, no reason why it should not alter the extra parameter a_l. In the following section we shall see that these operators may all be expressed in terms of I_2, I_3 and the O(3) scalar operators O_l^0 and Q_l^0; it is for this reason that they are so important. Many relations connecting the matrix elements of these product operators follow from equations (23); we shall calculate a typical one of these and be content simply to state the rest.

The first set of relationships connect the matrix elements of $O_l^{\pm i}$ to those of $O_{l\pm 1}^{\mp i}O_l^{\pm i}$: consider $\langle l, a_l, m|O_{l+1}^{-1}O_l^{+1}|l, a_l, m\rangle$ and insert a complete set of states between O_{l+1}^{-1} and O_l^{+1} excluding, however, those which give rise to zero matrix elements. This yields

$$\begin{aligned}\langle l, a_l, m|O_{l+1}^{-1}O_l^{+1}|l, a_l, m\rangle &\\ &= \sum_{b_{l-1}} \langle l, a_l, m|O_{l+1}^{-1}|l+1, b_{l+1}, m\rangle\langle l+1, b_{l+1}, m|O_l^{+1}|l, a_l, m\rangle \\ &= \sum_{b_{l+1}} \langle l+1, b_{l+1}, m|(O_{l+1}^{-1})^\dagger|l, a_l, m\rangle^*\langle l+1, b_{l+1}, m|O_l^{+1}|l, a_l, m\rangle.\end{aligned}$$

Using equation (23) and the reality of $\alpha_{1,l}$, we obtain

$$\begin{aligned}\langle l, a_l, m|O_{l+1}^{-1}O_l^{+1}|l, a_l, m\rangle &\\ &= \sum_{b_{l+1}} \left(\frac{1}{\alpha_{1,l}}\langle l+1, b_{l+1}, m|O_l^{+1}|l, a_l, m\rangle^*\right)\langle l+1, b_{l+1}, m|O_l^{+1}|l, a_l, m\rangle \\ &= \frac{1}{\alpha_{1,l}}\sum_{b_{l+1}} |\langle l+1, b_{l+1}, m|O_l^{+1}|l, a_l, m\rangle|^2.\end{aligned}$$

In a similar manner we obtain seven more relations summarized by the equations

$$\langle l, a_l, m|O_{l+i}^{-i}O_l^{+i}|l, a_l, m\rangle = \frac{1}{\alpha_{i,l}} \sum_{b_{l+i}} |\langle l+i, b_{l+i}, m|O_l^{+i}|l, a_l, m\rangle|^2$$

$$= \alpha_{i,l} \sum_{b_{l+i}} |\langle l, a_l, m|O_{l+i}^{-i}|l+i, b_{l+i}, m\rangle|^2 \tag{28}$$

$$\langle l, a_l, m|O_{l-i}^{+i}O_l^{-i}|l, a_l, m\rangle \doteq \alpha_{i,l-i} \sum_{b_{l-i}} |\langle l-i, b_{l-i}, m|O_l^{-i}|l, a_l, m\rangle|^2$$

$$= \frac{1}{\alpha_{i,l-i}} \sum_{b_{l-i}} |\langle l, a_l, m|O_{l-i}^{+i}|l-i, b_{l-i}, m\rangle|^2. \tag{29}$$

The next set of relations, which inter-relate the matrix elements of the $O_{l\pm i}^{\mp i}O_l^{\pm i}$, etc, do not depend on hermiticity properties, and are obtained by the insertion of a complete set of states between two shift operators. They are summarized by the equations

$$\sum_{a_l} \langle l, a_l, m|O_{l+i}^{-i}O_l^{+i}|l, a_l, m\rangle = \sum_{b_{l+i}} \langle l+i, b_{l+i}, m|O_l^{+i}O_{l+i}^{-i}|l+i, b_{l+i}, m\rangle \tag{30}$$

and

$$\sum_{a_l} \langle l, a_l, m|O_{l\pm1}^{\mp1}O_{l\pm2}^{\mp1}O_l^{\pm2}|l, a_l, m\rangle = \sum_{b_{l\pm1}} \langle l\pm1, b_{l\pm1}, m|O_{l\pm2}^{\mp1}O_l^{\pm2}O_{l\pm1}^{\mp1}|l\pm1, b_{l\pm1}, m\rangle$$

$$= \sum_{c_{l\pm2}} \langle l\pm2, c_{l\pm2}, m|O_l^{\pm2}O_{l\pm1}^{\mp1}O_{l\pm2}^{\mp1}|l\pm2, c_{l\pm2}, m\rangle. \tag{31}$$

The final set of relationships give the hermiticity properties of the product operators and are obtained by inserting a complete set of states between shift operators and using equation (23); they are

$$\langle l, a_l, m|O_{l\pm i}^{\mp i}O_l^{\pm i}|l, a_l', m\rangle = \langle l, a_l', m|O_{l\pm i}^{\mp i}O_l^{\pm i}|l, a, m\rangle^* \tag{32}$$

which show that the $O_{l\pm i}^{\mp i}O_l^{\pm i}$ are all hermitian operators,

$$\langle l, a_l, m|O_{l\pm1}^{\mp1}O_{l\pm2}^{\mp1}O_l^{\pm2}|l, a_l', m\rangle = \langle l, a_l', m|O_{l\pm2}^{\mp2}O_{l\pm1}^{\pm1}O_l^{\pm1}|l, a_l, m\rangle^* \tag{33}$$

and

$$\langle l, a_l, m|O_{l+1}^{-1}O_{l-1}^{+2}O_l^{-1}|l, a_l', m\rangle = \langle l, a_l, m|O_{l-1}^{+1}O_{l+1}^{-2}O_l^{+1}|l, a_l, m\rangle^*. \tag{34}$$

These show that $O_{l\pm1}^{\mp1}O_{l\pm2}^{\mp1}O_l^{\pm2}$ and $O_{l\pm2}^{\mp2}O_{l\pm1}^{\pm1}O_l^{\pm1}$ are hermitian conjugates, as are $O_{l+1}^{-1}O_{l-1}^{+2}O_l^{-1}$ and $O_{l-1}^{+1}O_{l+1}^{-2}O_l^{+1}$.

4. Explicit forms of the product operators

The product operators $O_{l\pm i}^{\mp i}O_l^{\pm i}$ and $O_{l\pm1}^{\mp1}O_{l\pm2}^{\mp1}O_l^{\pm2}$, etc, introduced in the last section commute with L^2 and l_0 and must therefore be expressible in terms of I_2, I_3, O_l^0 and Q_l^0. These expressions involve the l and m values of the states upon which they act, the dependence on m being due to the fact that the product operators do not commute with $l_\pm$. In order, however, to considerably simplify their calculations we restrict our considerations to the case when they act on states of zero m value. This is always permissible since it is already well known (Elliot 1958a, b, Bargmann and Moshinsky 1960, 1961 and De Baenst-van den Broucke *et al* 1970) that only integral values, for which there is always an $m = 0$ state, occur in the decomposition of representations of SU(3) into those of its O(3) subgroup. Also, as will be seen in the following paper, this seemingly drastic condition will not seriously detract from the generality of subsequent calculations since the eigenvalues of O_l^0 and Q_l^0 are independent of m.

The expressions for the product operators are, even with the above simplification, extremely tedious to calculate, so we shall merely state them. They are

$$\begin{aligned}\frac{O^{-1}_{l+1}O^{+1}_{l}}{(l+1)^2} = {} & -\tfrac{1}{9}(O^0_l)^2-\tfrac{1}{3}(l+1)(l+3)Q^0_l+24l(l+1)^2(2l+3)I_2 \\ & -2l(l+1)^2(2l^3+8l^2+12l+27)\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{O^{+1}_{l-1}O^{-1}_{l}}{l^2} = {} & -\tfrac{1}{9}(O^0_l)^2-\tfrac{1}{3}l(l-2)Q^0_l+24l^2(l+1)(2l-1)I_2 \\ & -2l^2(l+1)(2l^3-2l^2+2l-21)\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{O^{-2}_{l+2}O^{+2}_{l}}{(l+1)^2(l+2)^2} = {} & \tfrac{1}{9}(O^0_l)^2+\tfrac{1}{3}(2l+3)(2l+5)Q^0_l+24(l+1)(l+4)(2l+3)^2I_2 \\ & -2l(l+1)(2l+3)(2l^3+25l^2+3l+3)\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{O^{+2}_{l-2}O^{-2}_{l}}{l^2(l-1)^2} = {} & \tfrac{1}{9}(O^0_l)^2+\tfrac{1}{3}(2l-1)(2l-3)Q^0_l+24l(l-3)(2l-1)^2I_2 \\ & -2l(l+1)(2l-1)(2l^3-19l^2+9l+27)\end{aligned} \tag{38}$$

$$\begin{aligned}&\frac{9O^{-1}_{l+1}O^{-1}_{l+2}O^{+2}_{l}}{(l+1)^2(l+2)^2} \\ &\quad = -\tfrac{1}{3}(O^0_l)^3-3(l^2+5l+5)Q^0_lO^0_l+\tfrac{1}{4}(2l+3)(l+4)(l+5)[Q^0_l,O^0_l] \\ &\qquad -432(l+1)(l+2)(2l+3)I_2O^0_l+1296\sqrt{6}l(l+1)^2(l+4)(2l+3)^2I_3 \\ &\qquad -18l(l+1)(2l^4+12l^3+28l^2+54l+45)O^0_l\end{aligned} \tag{39}$$

$$\begin{aligned}&\frac{9O^{+1}_{l-1}O^{+1}_{l-2}O^{-2}_{l}}{l^2(l-1)^2} \\ &\quad = -\tfrac{1}{3}(O^0_l)^3-3(l^2-3l+1)Q^0_lO^0_l-\tfrac{1}{4}(2l-1)(l-3)(l-4)[Q^0_l,O^0_l] \\ &\qquad +432l(l-1)(2l-1)I_2O^0_l+1296\sqrt{6}l^2(l+1)(l-3)(2l-1)^2I_3 \\ &\qquad -18l(l+1)(2l^4-4l^3+4l^2-26l+9)O^0_l\end{aligned} \tag{40}$$

$$\begin{aligned}&\frac{9O^{-1}_{l+1}O^{+2}_{l-1}O^{-1}_{l}}{l^2(l+1)^2} \\ &\quad = -\tfrac{1}{3}(O^0_l)^3-3(l^2+l+1)Q^0_lO^0_l+\tfrac{1}{4}l(l+5)(2l-1)[Q^0_l,O^0_l] \\ &\qquad +432l(l+1)I_2O^0_l+1296\sqrt{6}l^2(l+1)^2(2l+3)(2l-1)I_3 \\ &\qquad -18l(l+1)(2l^4+4l^3+12l^2+10l+9)O^0_l\end{aligned} \tag{41}$$

$$\begin{aligned}&\frac{9O^{+1}_{l-1}O^{-2}_{l+1}O^{+1}_{l}}{l^2(l+1)^2} \\ &\quad = -\tfrac{1}{3}(O^0_l)^3-3(l^2+l+1)Q^0_lO^0_l-\tfrac{1}{4}(l+1)(l-4)(2l+3)[Q^0_l,O^0_l] \\ &\qquad +432l(l+1)I_2O^0_l+1296\sqrt{6}l^2(l+1)^2(2l+3)(2l-1)I_3 \\ &\qquad -18l(l+1)(2l^4+4l^3+12l^2+10l+9)O^0_l\end{aligned} \tag{42}$$

$$\frac{9O_{l-2}^{+2}O_{l-1}^{-1}O_l^{-1}}{l^2(l-1)^2}$$
$$\begin{aligned}= & -\tfrac{1}{3}(O_l^0)^3-3(l^2-3l+1)Q_l^0O_l^0+\tfrac{1}{4}l(l+1)(2l-5)[Q_l^0,O_l^0]\\ & +432l(l-1)(2l-1)I_2O_l^0+1296\sqrt{6}l^2(l+1)(l-3)(2l-1)^2I_3\\ & -18l(l+1)(2l^4-4l^3+4l^2-26l+9)O_l^0\end{aligned} \tag{43}$$

$$\frac{9O_{l+2}^{-2}O_{l+1}^{+1}O_l^{+1}}{(l+1)^2(l+2)^2}$$
$$\begin{aligned}= & -\tfrac{1}{3}(O_l^0)^3-3(l^2+5l+5)Q_l^0O_l^0-\tfrac{1}{4}l(l+1)(2l+7)[Q_l^0,O_l^0]\\ & -432(l+1)(l+2)(2l+3)I_2O_l^0+1296\sqrt{6}l(l+1)^2(l+4)(2l+3)^2I_3\\ & -18l(l+1)(2l^4+12l^3+28l^2+54l+45)O_l^0.\end{aligned} \tag{44}$$

The only nonhermitian operators appearing in these expressions are $[Q_l^0, O_l^0]$ and $Q_l^0O_l^0$, so the $O_{l\pm i}^{\mp i}O_l^{\pm i}$ are clearly hermitian, in agreement with equation (32). The triple product operators are not hermitian, and in fact equations (33) and (34) were used to obtain equations (42)–(44) from equations (39)–(41). Using the fact that O_l^{+i} and O_l^{-i} go over into each other on replacing l by $-(l+1)$, equations (36), (38) and (40) may easily be derived from, respectively, equations (35), (37) and (39).

Finally, from equations (35)–(38) we may derive the following formulae giving Q_l^0, $(O_l^0)^2$ and $O_{l-i}^{+i}O_l^{-i}$ in terms of $O_{l+i}^{-i}O_l^{+i}$:

$$Q_l^0=\frac{1}{(l+2)^2}\left(\frac{O_{l+1}^{-1}O_l^{+1}}{(l+1)^2}+\frac{O_{l+2}^{-2}O_l^{+2}}{(l+1)^2(l+2)^2}\right)-6(l+1)\{12(2l+3)I_2-l(2l^2+14l+3)\} \tag{45}$$

$$\begin{aligned}(O_l^0)^2= & \frac{-3(2l+3)(2l+5)}{(l+2)^2(l+1)^2}O_{l+1}^{-1}O_l^{+1}-\frac{3(l+3)}{(l+1)(l+2)^4}O_{l+2}^{-2}O_l^{+2}\\ & +18(l+1)^2(2l+3)^2\{12I_2-l(l+4)\}\end{aligned} \tag{46}$$

$$\begin{aligned}\frac{O_{l-1}^{+1}O_l^{-1}}{l^2}= & \frac{(l+5)}{(l+1)(l+2)^2}O_{l+1}^{-1}O_l^{+1}+\frac{(2l+1)}{(l+1)^2(l+2)^4}O_{l+2}^{-2}O_l^{+2}\\ & -24(l+1)^2(2l+1)\{9I_2-l(l+3)\}\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{O_{l-2}^{+2}O_l^{-2}}{l^2(l-1)^2}= & -\frac{4(2l+1)}{(l+1)^2(l+2)^2}O_{l+1}^{-1}O_l^{+1}+\frac{l(l-4)}{(l+1)^2(l+2)^4}O_{l+2}^{-2}O_l^{+2}\\ & +48l(2l+1)\{18I_2+l(l+1)(l-5)\}.\end{aligned} \tag{48}$$

These formulae will prove to be extremely useful in the following paper in obtaining the eigenvalues of O_l^0 and Q_l^0.

Acknowledgments

I should like to thank Professor A J Coleman for his hospitality in the Mathematics Department of Queen's University, Kingston, Ontario, during the academic year 1969–70, and to the National Research Council of Canada for a maintenance grant. The whole of this work was performed during that period.

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